Volumes and Other Invariants of Hyperbolic 3-Manifolds

by
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Low-dimensional topology is one of most active areas of mathematics now. The Poincare conjecture, proved by Perelman, was in the center of discussions for last few years. In some sense it played a big role in turning public attention to mathematics.

The Poincare conjecture can be considered as a special case of the Thurston conjecture formulated in 1978. William Thurston conjectured that any three-dimensional manifold can be cut in geometrical pieces, i.e. in pieces admitting one of eight three-dimensional geometries: Euclidean, spherical, hyperbolic, nilpotent, etc.

In the lectures we will discuss the most interesting of the eight Thurston geometries – the hyperbolic one. The theory of hyperbolic three-dimensional manifolds is well-developed part of geometric topology now. This theory combines topological and geometric methods. One of the fundamental results here is Mostow's rigidity theorem which states that two homeomorphic hyperbolic 3-manifolds are isometric. Hence, for hyperbolic 3-manifolds any geometric invariant is topological invariant too. One of such invariants is volume of a manifold. Volumes of hyperbolic 3-manifolds will be the main line of our lectures.

We will start with models of the hyperbolic space; discussing geodesics and isometries. As preliminary results we will present theorems about existence of polyhedra in the hyperbolic space. Following classical papers by Lobachevsky and Schlafli we will present two approaches for volume computations.

Next, we will discuss geometric part of the knot theory. It is known that most of knots in the 3-sphere are such that their complements admit hyperbolic geometry structures. This point of view admits one to introduce hyperbolic volumes of knot complements as very useful topological invariants of knots. We will discuss this approach considering the figure-eight knot as an example. Also, we will discuss relations of this invariant with other invariants of knots. In particular, we will discuss Kashaev conjecture relating hyperbolic volumes and values of Jones polynomials.

Finally, we will present explicit volume formulae for various infinite families of hyperbolic 3-manifolds. We will demonstrate how volume formulae can be used to study isometries of manifolds and their complexity.

Lecture 1: **Hyperbolic polyhedra: existence and volumes**

A. Models and isometries of the hyperbolic 3-space
B. Polyhedra in the hyperbolic 3-space, Andreev theorems
C. The Lobachevsky function
D. Volume behavior under deformations

Lecture 2: **Hyperbolic knots and their invariants**

A. Hyperbolic knots and links
B. Constructing of knot complements
C. Kashaev conjecture

Lecture 3: **Volumes of hyperbolic manifolds and orbifolds**

A. Poincare theorem on fundamental polyhedra
B. Weber-Seifert dodecahedral hyperbolic space
C. Loebell manifolds: volumes, isometries, complexity
D. Fibonacci manifolds: volumes, isometries, complexity