Topology of Spaces with Geometric Structure

by

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One of the fundamental problems in geometric topology, and in mathematics in general, is to identify mathematical spaces as manifolds. Manifolds have a rich structure that can be exploited in many applications. In fact, manifolds are the space of choice in which we model the fabric of our universe. Topological manifolds of dimension n are spaces that are locally homeomorphic to Euclidean n-space. Although easy to define, verifying that a space is a manifold is generally a formidable task.

Of particular interest are topological manifolds which have strong metric structures, such as Riemannian manifolds and Finsler spaces. Such metric structures are not inherent to topological manifolds, and imply certain smoothness features. In the 1940's Herbert Busemann sought to develop a basic set of axioms from which many of the fundamental structures of these types of spaces could still be obtained. To this end, he defined what he termed as a G-space, and what we now call a Busemann G-space. A Busemann G-space is a metric space that satisfies these four simple metric properties:

1. There is a point between any two points.
2. Every bounded infinite set has an accumulation point
3. If x and y are points of the space sufficiently close, there is a point z such that x-y-z (y is between x and z).
4. If x-y-z and x-y-w and yz=yw, then z=w.

The fact that this basic list of properties might even imply a topological manifold is quite remarkable. However, Busemann was able to prove that this is indeed true for spaces of dimensions n=1,2 in his initial writing in 1942. Busemann then conjectured that all Busemann G-spaces are topological manifolds, but remarked that the higher dimension cases were quite inaccessible at the time of his writing. This was an insightful comment. It wasn't until 1968 that the conjecture was proved by Krakus in the case of dimension n=3 and in 1993 that the conjecture was proved by Paul Thurston in the case of dimension n=4. Indeed the case of n=4 required the full power of modern developments in the area of manifolds recognition that had only begun in the mid to late 1970s. Berestovskii has also shown the conjecture to be true in the special case that Aleksandrov curvature is bounded below or above (1994 and 2002). However, the conjecture remains unsolved in the general case in higher dimensions.

This presentation will be an overview of the Busemann conjecture, progress that has been made, and why this 70 year old problem is a very important problem in both the areas of analytic geometry and topology.

Lecture 1:

A. Background and motivation behind the study of Busemann G-spaces
B. Geometric properties of Busemann G-spaces
C. Early results concerning the geometry of Busemann G-spaces
D. The Busemann Conjecture

Lecture 2:

A. Characterizations of manifolds
B. Recognition of manifolds
C. The Busemann Conjecture in Dimension 4

Lecture 3:

A. The Bing-Borsuk Problem
B. G-homogeneous G-spaces
C. Manifold factors
D. The Moore Problem
E. The Busemann Conjecture in Dimension n>4