Denis Saveliev: On ultrafilter extensions of first-order models

Abstract:

We show that any first-order model $\mathfrak{A}$ extends, in a canonical way, to a model $\beta\mathfrak{A}$ (of the same language) which consists of all ultrafilters over it and carries the standard compact Hausdorff topology on the set of ultrafilters. The extension procedure preserves relationships between models: any homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$ extends to a continuous homomorphism of $\beta\mathfrak{A}$ into $\beta\mathfrak{B}$. Moreover, if a model $\mathfrak{C}$ carries a compact Hausdorff topology which is compatible, in a sense, with its structure, then any homomorphism of $\mathfrak{A}$ into $\mathfrak{C}$ extends to a continuous homomorphism of $\beta\mathfrak{A}$ into $\mathfrak{C}$; analogous statements are true also for embeddings and some other types of relationships between models. Thus the construction provides a natural generalization of the Stone–Čech compactification of a discrete space to the case when the space is endowed with a first-order structure.