Izrek & domneva Hausmanna

> Kratek članek:
> A counter-example to Hausmann conjecture

I. Uvod:

X geodezičen metričen prostor

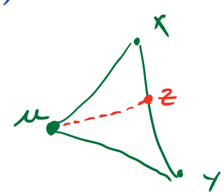
$r(X) \geq 0$ infimum števil r, ki zadoščajo:

ⓐ $\forall x,y \in X$, $d(x,y) < 2r$ $\exists!$ geodetka x do y $\quad < r$

ⓑ $\forall x,y,z,u \in X$ $\quad d(x,y), d(u,x), d(u,y) < r$

$\forall z$ na geodetki x do y:

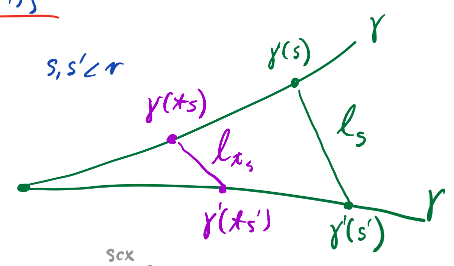
$$d(u,z) \leq \max\{d(u,x), d(u,y)\}$$



ⓒ $\forall \gamma, \gamma'$ geodetki parametrizirani $\quad s, s' < r$

z naravnim parametrom, $t \in [0,1]$:

$$\ell_{ts} \leq \ell_s$$



SCX ↘ $Rips(X,r) = \{\sigma \subset X;\ diam(\sigma) < r\}$

Izrek [Hausmann, 95]: $r(X) > 0 \Rightarrow \exists \varepsilon > 0:\ Rips(X,r) \simeq X,\ \forall r < \varepsilon.$

Domneva [H. 95]: Povezanost $Rips(X,r)$ je naraščajoča funkcija r za komp. skl. MNT.

II. Dokaz izreka

$\mathcal{U}$ pokritje X

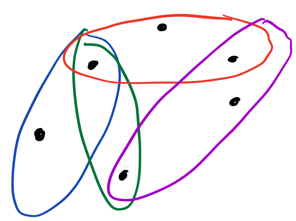
Živec

$N(\mathcal{U}) = \{\sigma \subset \mathcal{U};\ \bigcap_{U \in \sigma} U \neq \emptyset\}$

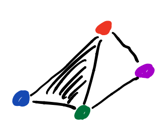
SCX

Vietoris

$V(\mathcal{U}) = \{\sigma \subset X;\ \exists U \in \mathcal{U}: \sigma \subset U\}$

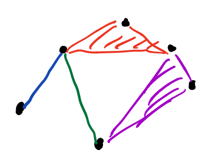


$N(\mathcal{U})$



$\simeq$ Dowker dualnost

$V(\mathcal{U})$



Izrek [Živec]: $\mathcal{U}$ pokritje X: (i) Vsi preseki končnih poddružin iz $\mathcal{U}$ so $\emptyset$ ali $\simeq \bullet$.

TAMMO TOM DIECK

(ii) Notranjost elementov $\mathcal{U}$ pokrije X.

Tedaj: $X \simeq N(\mathcal{U})$.

Če več: za $\mathcal{U} \prec \mathcal{U}'$ velja

$$\begin{tikzcd} X \arrow[r, "\simeq"] \arrow[d, equal] \arrow[dr, phantom, "\simeq"] & \mathcal{N}(\mathcal{U}) \arrow[r, "\simeq"] \arrow[d] \arrow[dr, phantom, "\simeq"] & \mathcal{V}(\mathcal{U}) \arrow[d] \\ X \arrow[r, "\simeq"] & \mathcal{N}(\mathcal{U}') \arrow[r, "\simeq"] & \mathcal{V}(\mathcal{U}') \end{tikzcd}$$

Opazka: Rips ni živec, je pa Vietorisov kompleks.

X geod., $r(X) > 0$. $r > 0$.

Dokaz: $\sigma \overset{\text{konč.}}{\subset} X$ $\quad$ $\operatorname{diam} \sigma < r$

$\Rightarrow \underbrace{\operatorname{diam}(\operatorname{conv}(\sigma)) < r}_{\text{zaradi (b)}}$, $\underbrace{\operatorname{conv}(\sigma) \simeq \bullet}_{\text{zaradi (c)}}$, $\boxed{\text{Vsaka } A \subset X,\ A \text{ konveksna},\ \operatorname{diam} A < r/2 \Rightarrow A \simeq \bullet}$

$\Rightarrow \mathcal{U} = \{\operatorname{conv}(\sigma);\ \sigma \subset X,\ \operatorname{diam} \sigma < r\}$ zadošča (i), $\mathcal{V}(\mathcal{U}) = \operatorname{Rips}(X, r)$.

X končno-dim MNT

v splošnem lahko $\operatorname{conv}(\sigma)$ induktivno širimo

$X \underset{\text{živec}}{\simeq} \mathcal{N}(\mathcal{U}) \underset{\text{Dowker}}{\simeq} \mathcal{V}(\mathcal{U}) = \operatorname{Rips}(X, r)$ ■

III. Protiprimer domneve

Izrek [Ferry, Okun, 95]: M gladka skl. MNT dimenzije vsaj 3. X geodezična. $f: M \twoheadrightarrow X$, $f^{-1}(x)$ povezan $\forall x \in X$. Tedaj obstaja zaporedje ψ_i Riem. metrik na M: $(M, \psi_i) \xrightarrow[GH]{i \to \infty} X$.

Izrek [Stabilnost vzt. diagramov]: $d_{GH}(X, Y) < \varepsilon \Rightarrow d_B(PD(X), PD(Y)) < 2\varepsilon$

Bottleneck dist. $\qquad$ vzt. diagrama preko Rips-f.

Izrek [Protiprimer]: $\exists$ Riem. m. ψ na S^3: (S^3, ψ) ne zadošča domnevi Haus.

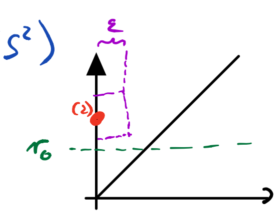
Dokaz: Izrek FO + Hopf: $S^3 \to S^2 \Rightarrow \exists \psi_i$ Riem. m. na S^3: $(S^3, \psi_i) \xrightarrow[GH]{i \to \infty} S^2$

Izberemo i: za $\psi = \psi_i$

$PD(S^3, \psi)$ ima točko v

$\exists r > 0$: $H_2(\operatorname{Rips}((S^3, \psi), r)) \neq 0$

$PD(S^2)$



Po UCT $H_2(\mathrm{Rips}((S^3,\psi),r)),\mathbb{Z}) \neq 0$

Whitehead $\Downarrow$ $\pi_1\, \mathrm{Rips}((S^3,\psi),r)$ trivialna

$$H_2(\mathrm{Rips}((S^3,\psi),r)) \cong \pi_2(\mathrm{Rips}(S^3,\psi),r) \neq 0. \quad \blacksquare$$

[Po Hausmannu vemo, da je $\pi_2(\mathrm{Rips}(S^3,\psi))=0$ za majhne r].

Povzetek o vztrajnostnih diagramih (PD).

X metričen

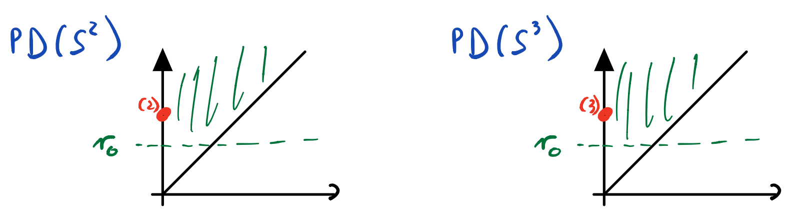
$\Downarrow$

Rips. filtracija $\{\mathrm{Rips}(X,r)\}_{r>0}$ $\quad \cdots \hookrightarrow \mathrm{Rips}(X,r_1) \hookrightarrow \mathrm{Rips}(X,r_2) \hookrightarrow \cdots$

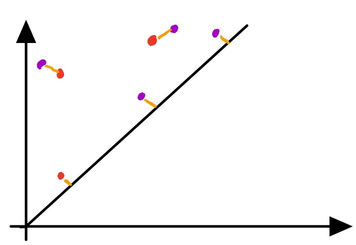
$p \in \{0,1,\dots\}$ $\quad \Downarrow$ $\mathbb{F}$ polje koef.

Vztr. modul $\underbrace{\{H_p(\mathrm{Rips}(X,r))\}_{r>0}}_{\cong}$ $\quad \cdots \to H_p(\mathrm{Rips}(X,r_1)) \to H_p(\mathrm{Rips}(X,r_2)) \to \cdots$

Primera:



Ozkogrlna razdalja d_B na diagramih



$$d_B(A,B) = \min_{\text{parjenja}} \max\{d_\infty(/)\}$$